

ORTHOGONAL APARTMENTS IN HILBERT GRASSMANNIANS. FINITE-DIMENSIONAL CASE

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ABSTRACT. Let H be a complex Hilbert space of finite dimension $n \geq 3$. Denote by $\mathcal{G}_k(H)$ the Grassmannian consisting of k -dimensional subspaces of H . Every orthogonal apartment of $\mathcal{G}_k(H)$ is defined by a certain orthogonal base of H and consists of all k -dimensional subspaces spanned by subsets of this base. For $n \neq 2k$ (except the case when $n = 6$ and k is equal to 2 or 4) we show that every bijective transformation of $\mathcal{G}_k(H)$ sending orthogonal apartments to orthogonal apartments is induced by a unitary or conjugate-unitary operator on H . The second result is the following: if $n = 2k \geq 8$ and f is a bijective transformation of $\mathcal{G}_k(H)$ such that f and f^{-1} send orthogonal apartments to orthogonal apartments then there is a unitary or conjugate-unitary operator U such that for every $X \in \mathcal{G}_k(H)$ we have $f(X) = U(X)$ or $f(X)$ coincides with the orthogonal complement of $U(X)$.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let H be a complex Hilbert space of finite or infinite dimension. For every natural $k < \dim H$ we denote by $\mathcal{G}_k(H)$ the Grassmannian consisting of k -dimensional subspaces of H . For every orthogonal base of H the associated *orthogonal apartment* of $\mathcal{G}_k(H)$ consists of all k -dimensional subspaces spanned by subsets of this base. Orthogonal apartments of Hilbert Grassmannians were introduced in [8]. This notion comes from the theory of Tits buildings [11]. Grassmannians related to a building of type A_{n-1} are the Grassmannians $\mathcal{G}_k(V)$, $k \in \{1, \dots, n-1\}$ formed by k -dimensional subspaces of an n -dimensional vector space V (see [6, 9] for the details) and every apartment of $\mathcal{G}_k(V)$ consists of all k -dimensional subspaces spanned by subsets of a certain base of V .

Recall that two closed subspaces $X, Y \subset H$ are *compatible* if there exist closed subspaces X', Y' such that $X \cap Y, X', Y'$ are mutually orthogonal and

$$X = X' + (X \cap Y), \quad Y = Y' + (X \cap Y).$$

The concept of orthogonal apartment is interesting for the following reason: orthogonal apartments can be characterized as maximal subsets of mutually compatible elements of $\mathcal{G}_k(H)$ [8, Proposition 1]. Note that the compatibility relation is defined for any logic, i.e. a lattice with an addition operation known as the *negation*. In classical logics any two elements are compatible and quantum logics contain non-compatible elements.

Consider the logic $\mathcal{L}(H)$ whose elements are closed subspaces of H and the negation is the operation of orthogonal complementary. In the case when H is infinite-dimensional and separable, this logic is exploited in mathematical foundations of quantum theory (see, for example, [12]). Every automorphism of $\mathcal{L}(H)$

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is induced by an unitary or conjugate-unitary operator on H . It follows from [5, Theorem 2.8] that for every bijective transformation f of $\mathcal{L}(H)$ preserving the compatibility relation in both directions, i.e. f and f^{-1} send compatible elements to compatible elements, there is an unitary or conjugate-unitary operator U such that for every $X \in \mathcal{L}(H)$ we have $f(X) = U(X)$ or $f(X)$ coincides with the orthogonal complement of $U(X)$. Note that if closed subspaces X and Y are compatible then the orthogonal complement of X is compatible to Y .

If H is infinite-dimensional and f is a bijective transformation of $\mathcal{G}_k(H)$ such that f and f^{-1} send orthogonal apartments to orthogonal apartments (equivalently, f preserves the compatibility relation in both directions) then f is induced by an unitary or conjugate-unitary operator [8, Theorem 1]. In this paper, a similar result will be obtained for the case when H is finite-dimensional. As in [8], we will use *complementary subsets* of orthogonal apartments, but the arguments will be more complicated.

Theorem 1. *Suppose that $\dim H = n$ is finite and not less than 3. Let f be a bijective transformation of $\mathcal{G}_k(H)$ sending orthogonal apartments to orthogonal apartments and let $n \neq 2k$. We also require that k is not equal to 2 or 4 if $n = 6$. Then f is induced by an unitary or conjugate-unitary operator on H .*

In Theorem 1 we do not require that the inverse transformation sends orthogonal apartments to orthogonal apartments. If H is finite-dimensional and f is a bijective transformation of $\mathcal{G}_k(H)$ sending compatible elements to compatible elements then f transfers orthogonal apartments to orthogonal apartments. This is a simple consequence of the following facts: the class of orthogonal apartments coincides with the class of maximal subsets of mutually compatible elements of $\mathcal{G}_k(H)$, all orthogonal apartments in $\mathcal{G}_k(H)$ are of the same finite cardinality.

Theorem 2. *Suppose that $\dim H = 2k \geq 8$. Let f be a bijective transformation of $\mathcal{G}_k(H)$ such that f and f^{-1} send orthogonal apartments to orthogonal apartments, in other words, f preserves the compatibility relation in both directions. Then there is an unitary or conjugate-unitary operator U on H such that for every $X \in \mathcal{G}_k(H)$ we have $f(X) = U(X)$ or $f(X)$ coincides with the orthogonal complement of $U(X)$.*

Remark 1. Apartments preserving transformations of Grassmannians corresponding to buildings of classical types are described in [6]. Also, there is a characterization of isometric embeddings of Grassmann graphs in terms of “generalized” apartments [7, Chapter 5].

2. GRASSMANN GRAPHS

To prove Theorems 1 and 2 we need some properties of Grassmann graphs. Suppose that $\dim H = n$ is finite. The *Grassmann graph* $\Gamma_k(H)$, $1 < k < n - 1$ is the graph whose vertex set is $\mathcal{G}_k(H)$ and two k -dimensional subspaces are adjacent vertices of this graph if their intersection is $(k - 1)$ -dimensional. In what follows we say that two k -dimensional subspaces of H are *adjacent* if they are adjacent vertices of $\Gamma_k(H)$.

If S and N are subspaces of H such that $\dim S < k$ and $\dim N > k$ then we denote by $[S]_k$ and $\langle N \rangle_k$ the sets consisting of k -dimensional subspaces containing S and contained in N , respectively. There are precisely the following two types of maximal cliques in $\Gamma_k(H)$:

- the *stars* $[S]_k$, $S \in \mathcal{G}_{k-1}(H)$;
- the *tops* $\langle N \rangle_k$, $N \in \mathcal{G}_{k+1}(H)$.

See, for example, [7, Section 3.2].

Let \mathcal{A} be the orthogonal apartment of $\mathcal{G}_k(H)$ defined by an orthogonal base B . The restriction of $\Gamma_k(H)$ to \mathcal{A} is isomorphic to the Johnson graph $J(n, k)$. As above, we have two types of maximal cliques:

- the *stars* $\mathcal{A} \cap [S]_k$,
- the *tops* $\mathcal{A} \cap \langle N \rangle_k$,

where $S \in \mathcal{G}_{k-1}(H)$ and $N \in \mathcal{G}_{k+1}(H)$ are spanned by subsets of B . Stars and tops of \mathcal{A} contain precisely $n - k + 1$ and $k + 1$ elements, respectively.

By classical Chow's theorem [2], every automorphism of the graph $\Gamma_k(H)$ is induced by an invertible semilinear operator if $n \neq 2k$. In the case when $n = 2k$, the automorphism group of $\Gamma_k(H)$ is generated by the automorphisms induced by invertible semilinear operators and the mapping $X \rightarrow X^\perp$, where X^\perp is the orthogonal complement of X . Huang's result [4] (see also [7, Section 3.10]) says that every bijective transformation of $\mathcal{G}_k(H)$ sending adjacent elements to adjacent elements is an automorphism of $\Gamma_k(H)$.

3. COMPLEMENTARY SUBSETS IN ORTHOGONAL APARTMENTS

Let \mathcal{A} be the orthogonal apartment of $\mathcal{G}_k(H)$ defined by an orthogonal base $\{e_i\}_{i=1}^n$ and let $1 < k < n - 1$. For every $i \in \{1, \dots, n\}$ we denote by $\mathcal{A}(+i)$ and $\mathcal{A}(-i)$ the sets consisting of all elements of \mathcal{A} which contain e_i and do not contain e_i , respectively. For any distinct $i, j \in \{1, \dots, n\}$ we define

$$\mathcal{A}(+i, +j) := \mathcal{A}(+i) \cap \mathcal{A}(+j),$$

$$\mathcal{A}(+i, -j) := \mathcal{A}(+i) \cap \mathcal{A}(-j),$$

$$\mathcal{A}(-i, -j) := \mathcal{A}(-i) \cap \mathcal{A}(-j).$$

A subset of \mathcal{A} is called *inexact* if there is an orthogonal apartment distinct from \mathcal{A} and containing this subset. By [8, Lemma 1], every maximal inexact subset is

$$\mathcal{A}(+i, +j) \cup \mathcal{A}(-i, -j).$$

We say that a subset $\mathcal{C} \subset \mathcal{A}$ is *complementary* if $\mathcal{A} \setminus \mathcal{C}$ is a maximal inexact subset. Then

$$\mathcal{A} \setminus \mathcal{C} = \mathcal{A}(+i, +j) \cup \mathcal{A}(-i, -j)$$

and

$$\mathcal{C} = \mathcal{A}(+i, -j) \cup \mathcal{A}(+j, -i).$$

This complementary subset is denoted by \mathcal{C}_{ij} . Observe that $\mathcal{C}_{ij} = \mathcal{C}_{ji}$.

Remark 2. Suppose that \mathcal{A} is an arbitrary (not necessarily orthogonal) apartment of $\mathcal{G}_k(H)$. A subset of \mathcal{A} is *inexact* if there is an apartment distinct from \mathcal{A} and containing this subset. Every maximal inexact subset is of type $\mathcal{A}(+i, +j) \cup \mathcal{A}(-i)$ and the corresponding complimentary subset is $\mathcal{A}(+i, -j)$, see [7, Section 5.2].

Lemma 1. Let f be a bijective transformation of $\mathcal{G}_k(H)$ sending orthogonal apartments to orthogonal apartments and let \mathcal{A} be an orthogonal apartment of $\mathcal{G}_k(H)$. Then $\mathcal{C} \subset \mathcal{A}$ is a complementary subset if and only if $f(\mathcal{C})$ is a complementary subset of $f(\mathcal{A})$.

Proof. It is clear that f transfers inexact subsets of \mathcal{A} to inexact subsets of $f(\mathcal{A})$. An inexact subset is maximal if and only if it contains

$$\binom{n-2}{k-2} + \binom{n-2}{k}$$

elements. This implies that maximal inexact subsets of \mathcal{A} go to maximal inexact subsets of $f(\mathcal{A})$. Since \mathcal{A} and $f(\mathcal{A})$ have the same number of such subsets, \mathcal{X} is a maximal inexact subset of \mathcal{A} if and only if $f(\mathcal{X})$ is a maximal inexact subset of $f(\mathcal{A})$. This gives the claim. \square

Lemma 2. *Let X, Y be distinct elements of an orthogonal apartment $\mathcal{A} \subset \mathcal{G}_k(H)$. If $\dim(X \cap Y) = m$ then there are precisely*

$$c(m) = (k - m)^2 + m(n - 2k + m)$$

distinct complementary subsets of \mathcal{A} containing this pair.

Proof. Since $\dim(X \cap Y) = m$, we have $\dim(X + Y) = 2k - m$. Let $\{e_i\}_{i=1}^n$ be one of the orthogonal bases associated to \mathcal{A} . If the complementary subset \mathcal{C}_{ij} contains both X and Y then one of the following possibilities is realized:

- (1) one of e_i, e_j belongs to $X \setminus Y$ and the other to $Y \setminus X$,
- (2) one of e_i, e_j belongs to $X \cap Y$ and the other is not contained in $X + Y$.

There are precisely $(k - m)^2$ and $m(n - 2k + m)$ distinct $\mathcal{C}_{ij} = \mathcal{C}_{ji}$ satisfying (1) and (2), respectively. \square

Remark 3. In the case when H is infinite-dimensional, there is the following characterization of the orthogonality relation [8, Lemma 2]: two elements in an orthogonal apartment \mathcal{A} are orthogonal if and only if there is only a finite number of complementary subsets of \mathcal{A} containing this pair. This implies that every bijective transformation of $\mathcal{G}_k(H)$ preserving the class of orthogonal apartments in both directions preserves also the orthogonality relation in both directions (see [8] for the details). By [3, 10], the latter condition guarantees that the bijection is induced by an unitary or conjugate-unitary operator.

4. PROOF OF THEOREM 1

Suppose that $\dim H = n$ is finite and not less than 3. Let f be a bijective transformation of $\mathcal{G}_k(H)$ sending orthogonal apartments to orthogonal apartments.

4.1. Preliminary remarks. The mapping $X \rightarrow X^\perp$ is a bijection between $\mathcal{G}_k(H)$ and $\mathcal{G}_{n-k}(H)$. It transfers every orthogonal apartment of $\mathcal{G}_k(H)$ to the orthogonal apartment of $\mathcal{G}_{n-k}(H)$ defined by the same orthogonal base. Thus $X \rightarrow f(X^\perp)^\perp$ is a bijective transformation of $\mathcal{G}_{n-k}(H)$ sending orthogonal apartments to orthogonal apartments. If it is induced by an unitary or conjugate-unitary operator then f is induced by the same operator. For this reason it is sufficient to prove Theorem 1 only in the case when $k \leq n - k$.

Suppose that $k = 1$. Then f transfers orthogonal elements of $\mathcal{G}_1(H)$ to orthogonal elements. For any 2-dimensional subspace $Y \subset H$ we take orthogonal 1-dimensional subspaces $X_1, X_2 \subset Y$ and extend them to an orthogonal apartment $\{X_i\}_{i=1}^n$ in $\mathcal{G}_1(H)$. If X is a 1-dimensional subspace of Y then $f(X)$ is orthogonal to $f(X_i)$ for every $i \geq 3$, i.e. $f(X)$ is contained in $f(X_1) + f(X_2)$. So, f sends all lines of the projective space over H to subsets of lines and, by the Fundamental Theorem of Projective Geometry [1], f is induced by an invertible semilinear operator. This

operator transfers orthogonal vectors to orthogonal vectors. An easy verification shows that it is a scalar multiple of an unitary or conjugate-unitary operator U . It is clear that f is induced by U .

From this moment we will suppose that $1 < k \leq n - k$.

4.2. The case $n \neq 2k + 2$. Following Lemma 2, we consider the quadratic function

$$c(x) = (k - x)^2 + x(n - 2k + x) = 2x^2 - (4k - n)x + k^2.$$

It takes the minimal value on $x = \frac{4k-n}{4}$. This implies that

$$c(k - 1) > c(m)$$

for all $m \in \{0, 1, \dots, k - 2\}$ if

$$k - 1 > \frac{4k - n}{2}$$

or, equivalently, if

$$k < \frac{n - 2}{2}.$$

By our assumption, $n = 2k + l$ for some natural $l \geq 0$ and the latter inequality fails only in the case when $l \in \{0, 1, 2\}$.

If $n = 2k + 2$ then $k - 1$ is equal to $(4k - n)/2$ which means that

$$c(k - 1) = c(0)$$

and the latter number is greater than $c(m)$ for any $m \in \{1, \dots, k - 2\}$.

If $n = 2k + 1$ then $k - 1$ is less than $(4k - n)/2$ which implies that $c(k - 1) < c(0)$, but we have

$$c(k - 1) \neq c(m)$$

for every $m \in \{0, 1, \dots, k - 2\}$. Indeed, if $c(k - 1) = c(x)$ and $x \neq k - 1$ then an easy calculation shows that $x = 1/2$.

Lemma 2 together with the above arguments give the following characterization of adjacent elements in orthogonal apartments.

Lemma 3. *Suppose that n is not equal to $2k$ or $2k + 2$. Let \mathcal{A} be an orthogonal apartment of $\mathcal{G}_k(H)$. Then $X, Y \in \mathcal{A}$ are adjacent if and only if there are precisely $c(k - 1)$ distinct complementary subsets of \mathcal{A} containing this pair.*

Lemma 4. *As in the previous lemma, we suppose that n is not equal to $2k$ or $2k + 2$ and \mathcal{A} is an orthogonal apartment of $\mathcal{G}_k(H)$. Then $X, Y \in \mathcal{A}$ are adjacent if and only if $f(X)$ and $f(Y)$ are adjacent; moreover, f transfers stars of \mathcal{A} to stars of $f(\mathcal{A})$.*

Proof. Using Lemmas 1 and 3 we show that $X, Y \in \mathcal{A}$ are adjacent if and only if the same holds for $f(X)$ and $f(Y)$. Then f transfers every star of \mathcal{A} to a star or a top of $f(\mathcal{A})$. Stars and tops contain $n - k + 1$ and $k + 1$ elements, respectively. Since $n \neq 2k$, these numbers are distinct and the image of every star is a star. \square

We prove Theorem 1 for $n \neq 2k + 2$.

Let X and Y be adjacent elements of $\mathcal{G}_k(H)$ which are not contained in an orthogonal apartment. Denote by N the orthogonal complement of $X \cap Y$. The dimension of N is equal to $n - k + 1$. Let S be the intersection of $X + Y$ with N . This is a 2-dimensional subspace. Then

$$\dim(S^\perp \cap N) = n - k - 1 \geq 2$$

(since $n = 2k + l$ and $l \geq 1$, we have $n - k - 1 = k + l - 1 \geq k \geq 2$). This implies the existence of orthogonal 1-dimensional subspaces $P, Q \subset N$ which are orthogonal to S . We set

$$X' = P + (X \cap Y) \quad \text{and} \quad Y' = Q + (X \cap Y).$$

Then X, X', Y' are mutually compatible and the same holds for Y, X', Y' . Let \mathcal{A} and \mathcal{A}' be orthogonal apartments containing X, X', Y' and Y, X', Y' , respectively. Since X, X', Y' are contained in a star of \mathcal{A} , Lemma 4 guarantees that $f(X), f(X'), f(Y')$ are contained in a star of $f(\mathcal{A})$. Similarly, we establish that $f(Y), f(X'), f(Y')$ are in a star of $f(\mathcal{A}')$. Therefore, $f(X)$ and $f(Y)$ both contain the $(k-1)$ -dimensional subspace $f(X') \cap f(Y')$ which implies that they are adjacent.

So, f sends adjacent vertices of $\Gamma_k(H)$ to adjacent vertices which implies that f is an automorphism of $\Gamma_k(H)$. Then f is induced by an invertible semilinear operator. This operator sends orthogonal vectors to orthogonal vectors. Hence it is a scalar multiple of an unitary or conjugate-unitary operator U . The transformation f is induced by U .

4.3. The case $n = 2k + 2$. Suppose that $n = 2k + 2$ and consider an orthogonal apartment $\mathcal{A} \subset \mathcal{G}_k(H)$. By the previous subsection, if $X, Y \in \mathcal{A}$ and there are precisely $c(k-1)$ distinct complementary subsets of \mathcal{A} containing this pair then X and Y are adjacent or $\dim(X \cap Y) = 0$ and they are orthogonal.

We say that two distinct complementary subsets \mathcal{C}_{ij} and $\mathcal{C}_{i'j'}$ are *adjacent* if $\{i, j\} \cap \{i', j'\} \neq \emptyset$. In this case, we have

$$|\mathcal{C}_{ij} \cap \mathcal{C}_{i'j'}| = \binom{n-3}{k-1} + \binom{n-3}{k-2} = \binom{n-2}{k-1} = \binom{2k}{k-1};$$

otherwise, we get

$$|\mathcal{C}_{ij} \cap \mathcal{C}_{i'j'}| = 4 \binom{n-4}{k-2} = 4 \binom{2k-2}{k-2}.$$

The equality

$$\binom{2k}{k-1} = 4 \binom{2k-2}{k-2}$$

holds only for $k = 2$, i.e. $n = 6$. Therefore, for $n \neq 6$ two complementary subsets $\mathcal{C}, \mathcal{C}' \subset \mathcal{A}$ are adjacent if and only if $f(\mathcal{C})$ and $f(\mathcal{C}')$ are adjacent complementary subsets of $f(\mathcal{A})$.

If $X, Y \in \mathcal{A}$ are orthogonal then for every complementary subset $\mathcal{C} \subset \mathcal{A}$ containing this pair there is a complementary subset of \mathcal{A} containing X, Y and adjacent to \mathcal{C} . In the case when $X, Y \in \mathcal{A}$ are adjacent, there is the unique complementary subset of \mathcal{A} containing X, Y and non-adjacent to any other complementary subset of \mathcal{A} containing X, Y .

Using the latter observation, we establish the direct analogue of Lemma 4 for $n = 2k + 2 \neq 6$. As in the previous subsection, we show that f is induced by an unitary or conjugate-unitary operator.

Remark 4. Consider the case when $n = 6$ and $k = 2$. If \mathcal{A} is an orthogonal apartment of $\mathcal{G}_k(H)$ then any distinct $X, Y \in \mathcal{A}$ are contained in precisely 4 distinct complementary subsets of \mathcal{A} . The intersection of any two distinct complementary subsets consists of 3 elements. Thus the dimension of the intersection of $X, Y \in \mathcal{A}$ cannot be determined in terms of complementary subsets.

5. PROOF OF THEOREM 2

Suppose that $\dim H = 2k \geq 8$ and f is a bijective transformation of $\mathcal{G}_k(H)$ such that f and f^{-1} send orthogonal apartments to orthogonal apartments.

In this case we have

$$c(m) = (k - m)^2 + m^2.$$

It is easy to see that

$$c(0) > c(m)$$

for every $m \in \{1, \dots, k - 1\}$ and

$$c(m) = c(m')$$

only in the case when $m' = m$ or $m' = k - m$. The standard arguments give the following.

Lemma 5. *Let X, Y be distinct elements in an orthogonal apartment $\mathcal{A} \subset \mathcal{G}_k(H)$. If $\dim(X \cap Y) = m$ then $\dim(f(X) \cap f(Y))$ is equal to m or $k - m$.*

Since $\dim H = 2k$, the orthogonal complement X^\perp is the unique element of $\mathcal{G}_k(H)$ orthogonal to $X \in \mathcal{G}_k(H)$. Any pair X, X^\perp is contained in a certain orthogonal apartment of $\mathcal{G}_k(H)$. In the previous lemma we put $m = 0$ and get the following.

Lemma 6. *For every $X \in \mathcal{G}_k(H)$ we have $f(X^\perp) = f(X)^\perp$.*

Let \mathcal{G}' be the set of all 2-element subsets $\{X, X^\perp\} \subset \mathcal{G}_k(H)$. By Lemma 6, f induces a bijective transformation of \mathcal{G}' . We denote this transformation by f' .

Consider the graph Γ' whose vertex set is \mathcal{G}' and subsets

$$\{X, X^\perp\}, \{Y, Y^\perp\} \in \mathcal{G}'$$

are adjacent vertices of this graph if X is adjacent to Y or Y^\perp . The latter conditions guarantees that X^\perp is adjacent to Y^\perp or Y , respectively. Two elements of \mathcal{G}' will be called *adjacent* if they are adjacent vertices of Γ' .

For every $(k - 1)$ -dimensional subspace $S \subset H$ we denote by $\mathcal{C}(S)$ the set of all $\{X, X^\perp\} \in \mathcal{G}'$ such that X or X^\perp contains S (then the $(k + 1)$ -dimensional subspace S^\perp contains X^\perp or X , respectively). This is a maximal clique of the graph Γ' .

Lemma 7. *If \mathcal{C} is a maximal clique of Γ' then $\mathcal{C} = \mathcal{C}(S)$ for a certain $(k - 1)$ -dimensional subspace S .*

Proof. Let $\{X, X^\perp\} \in \mathcal{C}$. Denote by \mathcal{X} the set of all $Y \in \mathcal{G}_k(H)$ adjacent to X and such that $\{Y, Y^\perp\} \in \mathcal{C}$. It is clear that \mathcal{X} is a maximal clique of $\Gamma_k(H)$. If this is the star corresponding to a $(k - 1)$ -dimensional subspace S then $\mathcal{C} = \mathcal{C}(S)$. If \mathcal{X} is the top defined by a $(k + 1)$ -dimensional subspace N then N^\perp is $(k - 1)$ -dimensional and $\mathcal{C} = \mathcal{C}(N^\perp)$. \square

Lemma 8. *If X, Y, Z are mutually adjacent elements of $\mathcal{G}_k(H)$ contained in an orthogonal apartment then there is the unique maximal clique of Γ' containing the corresponding elements of \mathcal{G}' .*

Proof. For $S = X \cap Y \cap Z$ and $N = X + Y + Z$ one of the following possibilities is realized:

- (1) $\dim S = k - 1$ and $\dim N = k + 2$,
- (2) $\dim S = k - 2$ and $\dim N = k + 1$.

The required maximal clique is $\mathcal{C}(S)$ or $\mathcal{C}(N^\perp)$, respectively. \square

Lemma 9. *The transformation f' is an automorphism of the graph Γ' .*

Proof. Let $\{X, X^\perp\}$ and $\{Y, Y^\perp\}$ be adjacent elements of \mathcal{G}' . We need to show that f' transfers them to adjacent elements. It is sufficient to restrict ourself to the case when X and Y are adjacent.

Suppose that there is an orthogonal apartment of $\mathcal{G}_k(H)$ containing X and Y . Note that X^\perp and Y^\perp also belong to this apartment. By Lemma 5, $f(X)$ and $f(Y)$ are adjacent or

$$\dim(f(X) \cap f(Y)) = 1.$$

The latter equality implies that $f(X)$ is adjacent to $f(Y)^\perp = f(Y^\perp)$. Therefore, f' transfers $\{X, X^\perp\}$ and $\{Y, Y^\perp\}$ to adjacent elements of \mathcal{G}' .

Consider the case when there is no orthogonal apartment containing X and Y . Let N be the orthogonal complement of $X \cap Y$ and let S be the intersection of $X + Y$ with N . The dimensions of N and S are equal to $k + 1$ and 2, respectively. Also, we have

$$\dim(S^\perp \cap N) = k - 1 \geq 3.$$

This means that N contains three mutually orthogonal 1-dimensional subspaces P, Q, T which are orthogonal to S . We set

$$X' = P + (X \cap Y), \quad Y' = Q + (X \cap Y), \quad Z' = T + (X \cap Y).$$

Note that X, Y, X', Y', Z' are mutually adjacent. Since X, X', Y', Z' are mutually compatible, there is an orthogonal apartment of $\mathcal{G}_k(H)$ containing them. The same holds for Y, X', Y', Z' . Then there is an orthogonal apartment containing

$$f(X), f(X'), f(Y'), f(Z')$$

and, by the above arguments, the corresponding elements of \mathcal{G}' are mutually adjacent. Lemma 8 implies the existence of the unique maximal clique

$$\mathcal{C}(M), \quad M \in \mathcal{G}_{k-1}(H)$$

containing them. Similarly, there is the unique maximal clique

$$\mathcal{C}(N), \quad N \in \mathcal{G}_{k-1}(H)$$

which contains the elements of \mathcal{G}' corresponding to

$$f(Y), f(X'), f(Y'), f(Z').$$

Then $\mathcal{C}(M) \cap \mathcal{C}(N)$ contains at least 3 elements and Lemma 8 guarantees that $M = N$. So, the elements of \mathcal{G}' corresponding to $f(X)$ and $f(Y)$ belong to a certain maximal clique of Γ' , in other words, they are adjacent.

Similarly, we show that f^{-1} sends adjacent elements to adjacent elements. \square

It follows from Lemma 9 that f' sends maximal cliques of Γ' to maximal cliques and there is a transformation g of $\mathcal{G}_{k-1}(H)$ such that

$$f'(\mathcal{C}(S)) = \mathcal{C}(g(S))$$

for every $S \in \mathcal{G}_{k-1}(H)$. This transformation is bijective.

Lemma 10. *The transformation g sends orthogonal apartments to orthogonal apartments.*

Proof. Let B be an orthogonal base of H and let \mathcal{A} be the associated orthogonal apartment of $\mathcal{G}_k(H)$. We take any orthogonal base B' corresponding to the orthogonal apartment $f(\mathcal{A})$. An easy verification shows that g transfers the orthogonal apartment defined by B to the orthogonal apartment defined by B' . \square

By Theorem 1, the transformation g is induced by an unitary or conjugate-unitary operator U . This operator satisfies the required condition.

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